STABILITY OF A COMPRESSIBLE BOUNDARY LAYER RELATIVE TO A LOCALIZED DISTURBANCE

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The problem of stability of an incompressible boundary layer relative to a localized disturbance is considered in a linear approximation. It is shown that the stability analysis reduces to the study of a discrete spectrum of eigenvalues of the corresponding boundary value problem. By means of numerical integration, analysis of the character of the emerging instability is carried out for an unstable mode for the Mach number M = 4.5.

1. The problem of stability of a compressible boundary layer on a flat plate relative to a localized disturbance is considered. The plate is located in the xz plane parallel to the incident flow; the velocity of the incident flow is directed along the x axis. The equations of gas dynamics are linearized relative to a small deviation q from the stationary distribution Q_S:

$$Q(t) = Q_s + q(t) \tag{1.1}$$

For the system of equations that is linear relative to q, we solve the Cauchy problem and study the asymptotic behavior of the solution for $t \to \infty$. The analysis is carried out with the following assumptions. The Prandtl number (Pr) and the ratio of the specific heats (γ) are considered to be constant; the state equation of an ideal gas is used. The dependence of the stationary distribution of velocity, temperature, and density on x is neglected, and the flow in the nonperturbed boundary layer is assumed to be plane-parallel.

2. For the assumptions just made (which are usual for problems of stability of boundary layers [1, 2]), the solution of the problem with the initial data is set up by means of the Laplace transformation with respect to time and the Fourier transformation with respect to the x, z coordinates

$$q_*(y) = \frac{1}{(2\pi)^2}\int_0^\infty e^{-pt}\,dt \int q(x, y, z, t)\, e^{-i\mathbf{kr}}\, d\mathbf{r}$$
$$\mathbf{kr} = \alpha x + \beta z, \qquad d\mathbf{r} = dx\,dz \tag{2.1}$$

For the amplitude $q_*(y)$ we obtain a nonhomogeneous system of ordinary differential equations of the eighth order which conveniently is written in matrix form brought to the normal form

$$q_*' = A(y)\, q_* + \xi \tag{2.2}$$

The prime denotes differentiation with respect to y. The components of the matrix A and the vector ξ are presented in Sec. 5. The components q_1, q_4, q_5, q_3, q_7 of the vector q_* are the amplitude of the x, y, z disturbances – of the velocity, pressure, and temperature components, respectively:

$$q_2 = q_1', \quad q_6 = q_5', \quad q_8 = q_7'$$

The boundary conditions for (2.2) consist of the conditions that the velocity components become zero and that the temperature and heat flow at y=0 are continuous

$$q_1(0) = q_4(0) = q_5(0) = 0, \quad a q_8(0) + b q_7(0) = 0 \tag{2.3}$$

and the conditions of boundedness of the amplitudes for $y \to \infty$. By the method of variation of constants the general solution of Eq. (2.2) can be written in the form

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$$q_*(y) = W(y)(d + c(y)), \qquad c(y) = \int_0^y W^{-1}(\tau)\xi(\tau)\,d\tau \tag{2.4}$$

or for the components

$$q_n = \Sigma(d_i + c_i)w_{ni} \qquad (n = 1, \ldots, 8)$$
$$c_i = \int_0^y \frac{\det W_i(\tau)}{\det W(\tau)}\,d\tau \tag{2.5}$$

Here W(y) is a certain fundamental matrix of the homogeneous equation (2.2); W_i is a matrix obtained from W by replacing the i-th column by the vector ξ ; d is a constant vector determined from the boundary conditions.

We consider together with (2.2) the homogeneous "limiting" equation

$$\psi' = A(\infty)\psi \qquad (A(\infty) = \lim_{y\to\infty} A(y)) \tag{2.6}$$

and its fundamental matrix

$$\Psi(y) = \Phi(y)P$$

Here Φ is another fundamental matrix of Eq. (2.6):

$$\Phi = e^{yA(\infty)} = Pe^{yJ}P^{-1}$$

while P is a constant matrix which brings $A(\infty)$ into the canonical form J. The columns of Ψ form a system of linearly independent solutions of Eq. (2.6):

$$\psi_n = p_n e^{\lambda_n y} \qquad (n = 1, \ldots, 8)$$

where λ_n are the roots of the characteristic equation

$$\det(A(\infty) - \lambda E) = 0 \tag{2.7}$$

and E is a unit matrix.

It can be shown (Sec. 6) that the homogeneous equation (2.6) has solutions which asymptotically coincide for large y with ψ_n. In the role of the fundamental matrix W we take a matrix made up of these solutions; here w_j (j-th column of W) with $j \le 4$ are increasing, while with $j > 4$ they are decreasing solutions for $y \to \infty$. Then from the conditions of boundedness in the case of $y \to \infty$ it follows

$$d_i = \frac{1}{\det P}\int_0^\infty \det W_i(\tau)\exp\left[\int_\tau^\infty \operatorname{Sp} A(s)\,ds\right]d\tau \qquad (i \leqslant 4)$$

Here we have taken into account the fact that

$$\det W(y) = \det P \exp\left[-\int_y^\infty \operatorname{Sp} A(s)\,ds\right]$$

From the conditions (2.3) it follows that

$$d_i = \frac{\det G_{i-4}}{\det G} \qquad (i > 4)$$

$$G = \begin{pmatrix} w_{15}(0) & \ldots & w_{18}(0) \\ w_{45}(0) & \ldots & w_{48}(0) \\ w_{55}(0) & \ldots & w_{58}(0) \\ aw_{85}(0) + bw_{75}(0) & \ldots & aw_{88}(0) + bw_{78}(0) \end{pmatrix}$$

The matrix G_i is obtained from G by replacement of the i-th column by the vector η with the components

$$\eta_j = -\sum_{i=1}^{4} d_i z_{ji} \qquad (j = 1, 2, 3, 4)$$

$$Z = \begin{pmatrix} w_{11}(0) & \dots & w_{14}(0) \\ w_{41}(0) & \dots & w_{44}(0) \\ w_{51}(0) & \dots & w_{54}(0) \\ aw_{81}(0) + bw_{71}(0) & \dots & aw_{84}(0) + bw_{74}(0) \end{pmatrix}$$

We finally can write

$$d_i + c_i = \begin{cases} -\dfrac{1}{\det P} \displaystyle\int_y^{\infty} \exp\left[\int_\tau^{\infty} \mathrm{Sp}\, A(s)\, ds\right] \det W_i d\tau & (1 \leqslant i \leqslant 4) \\ \dfrac{1}{\det P} \displaystyle\int_0^{y} \exp\left[\int_\tau^{\infty} \mathrm{Sp}\, A(s)\, ds\right] \det W_i d\tau + \dfrac{\det G_{i-4}}{\det G} & (5 \leqslant i \leqslant 8) \end{cases}$$

The inverse Fourier and Laplace transformations applied to (2.5) give the solution of the Cauchy problem

$$q_n = \frac{1}{2\pi i} \int e^{ikr} dk \int_{s-i\infty}^{s+i\infty} e^{pt} \sum_{i=1}^{8} (d_i + c_i)\, w_{ni} dp \tag{2.8}$$

The path of integration inthe complex p-plane in the expression (2.8) passes on the right of all singularities of the function behind the integral sign. Use of the Laplace transformation (and, hence, the possibility of carrying out such a path) is considered justifiable from a physical viewpoint – a small disturbance cannot grow faster than an exponential function.

3. The asymptotic expansion of the solution (2.8) in the case of $t \to \infty$ is obtained if we close the contour of integration in the left half-plane of the p plane. The principal contribution to the expansion (in the case of instability) is provided by the singularities of the integrand expression located in the region $\mathrm{Re}\, p > 0$. The singularities can be of two types: branching points and poles. Singularities of the first type do not contribute to instability. Since w_j are analytic in D_j (see Sec. 6), it is sufficient to show that branching points of the roots of Eq. (2.7) are located in the half-plane $\mathrm{Re}\, p \leq 0$. The determinant in Eq. (2.7) can be found if we take into account the fact that Eq. (2.7) considered in relation to $\omega = p + i\alpha$, $v = k^2 - \lambda^2$ is dispersive for small disturbances in a gas at rest:

$$(\varepsilon v + \omega)^2 [\omega (\varepsilon v + 3\omega / 4)(\varepsilon v + \mathrm{Pr}\omega / \gamma) + 3v (\varepsilon v + \mathrm{Pr}\, \omega) / 4\gamma M^2] = 0 \tag{3.1}$$

Here $\varepsilon = 1/R$, R is the Reynolds number of the incident flow across the thickness of the boundary layer, M is the Mach number of the incident flow, and $k = \sqrt{\alpha^2 + \beta^2}$.

The coordinates of branching points $\lambda(p)$ are determined from the equations

$$\varepsilon k^2 + \omega = 0 \tag{3.2}$$

$$[(\mathrm{Pr} / \gamma + 3/4)\, \omega + 3\, \mathrm{Pr}/4\, \varepsilon\gamma M^2]^2 - 3\, \mathrm{Pr}\, \omega\, (\omega + 3/4\, \varepsilon\gamma M^2)/\gamma = 0 \tag{3.3}$$

$$\omega(\varepsilon k^2 + 3\omega/4)(\varepsilon k^2 + \mathrm{Pr}\omega/\gamma) + 3k^2(\varepsilon k^2 + \mathrm{Pr}\omega)/4\gamma M^2 = 0 \tag{3.4}$$

Solving these equations for ω, we obtain

$$\omega_1 = -\varepsilon k^2 \tag{3.5}$$

$$\omega_{2,3} = -\frac{3\mathrm{Pr}}{4\varepsilon\gamma^2 M^2} \frac{\mathrm{Pr} + 3/4\gamma - 3/2 \pm [3(\gamma - 1)(\mathrm{Pr} - 0.75)]^{1/2}}{(\mathrm{Pr}/\gamma - 0.75)^2} \tag{3.6}$$

$$\omega_{4,5} = -\frac{2}{3}\left(1 + \frac{3}{4}\frac{(\gamma - 1)}{\mathrm{Pr}}\right)\varepsilon k^2 \pm i\frac{k}{M} + o(\varepsilon) \tag{3.7}$$

$$\omega_6 = -\varepsilon k^2 / \mathrm{Pr} + o(\varepsilon) \tag{3.8}$$

The roots of Eq. (3.4) are located in the left half-plane independently of ε. This follows from the Hurwitz criterion [3].

Thus, instability is connected with the discrete spectrum of the boundary value problem (2.2), (2.3), and the principal part of the asymptotic ($t \to \infty$) expansion of the solution is written in the form

$$q_n = \int d\mathbf{k} e^{i\mathbf{kr}} \sum_m \mathrm{res}_m \left(e^{pt} \sum_{i=1}^{8} (d_i + c_i) w_{ni} \right) \tag{3.9}$$

The summation is carried out over the poles of the integrand expression in (2.8) located in the right half-plane (if there are no such poles, then obviously $q_n \to 0$ for $t \to \infty$). Since A, P, W, ξ are analytic in the region $p > 0$, the expression (3.9) can be rewritten as

$$q_n = \int d\mathbf{k} \sum_m e^{p_m t + i\mathbf{kr}} \left. \frac{F_n}{\partial F / \partial p} \right|_{p=p_m} \tag{3.10}$$

$$F_n = \sum_{i=5}^{8} w_{ni} \det G_{i-4}, \qquad F = \det G$$

where $p_m(\mathbf{k})$ are the roots of the equation

$$F(R, \mathbf{k}, p) = 0 \tag{3.11}$$

We know [1, 2] that for R that is greater than a certain R_c, Eq. (3.11) has roots which satisfy in some region of the $\mathbf{k}$-space the condition $\mathrm{Re}\, p > 0$ [the expression (3.10) is written for the case of simple roots]. This means that the disturbance, having the form of a monochromatic wave, for $R > R_c$ will increase without bounds with the time. However, the asymptotic behavior of a localized disturbance can differ from the behavior of monochromatic components and, in spite of the growth of some of them, the disturbance in a finite region of space will die out because of the disturbance being carried down the stream [4, 5] (convective instability). If, however, a localized disturbance grows with time within a finite region of space, then the flow is said to be absolutely unstable. The formulation of this problem is due to L. D. Landau and E. M. Lifshits [4]. The elucidation of the problem concerned with the character of the emerging instability reduces to the analysis of the asymptotic behavior of the integral in the expression (3.10) in the case $t \to \infty$. We interchange the order of integration and summation in (3.10) and consider one term of the resulting series

$$J_m = \int_{\Omega_m} d\mathbf{k} e^{p_m t + i\mathbf{kr}} \left. \frac{F_n}{\partial F / \partial p} \right|_{p=p_m} \tag{3.12}$$

integration is confined to the region Ω_m in which $\mathrm{Re}\, p_m \geq 0$.

The asymptotic behavior of the integral depends on the character of location of the level lines $\mathrm{Re}\, p_m =$ const in the complex k-plane. The possible cases are shown in Fig. 1 for two-dimensional ($\beta = 0$) disturbances (zero level lines are depicted). Figure 1a corresponds to the stable case; Fig. 1b corresponds to the case of convective instability, since we can place the path of integration into the complex plane so that along it $\mathrm{Re}\, p_m < 0$ and $J_m \to 0$ for $t \to \infty$ in the case of a fixed r.

In the third case (Fig. 1c) for the estimate of the integral we can use the cross-over method [6], placing the path of integration so that it passes through the saddle point in the direction of the steepest descent, and for the integral we obtain an estimate from which it follows that for $t \to \infty$ it grows without bounds as $\exp[\mathrm{Re}\, p_m(\alpha_0) t]/\sqrt{t}$ (α_0 is the saddle point). The cross-over method admits a direct generalization for the case of several variables [7], and can be applied to three-dimensional disturbances. The condition that absolute instability emerges for a certain $R = R_a$ is written in the form

$$\frac{\partial p_m}{\partial \alpha} = 0, \qquad \frac{\partial p_m}{\partial \beta} = 0, \qquad \mathrm{Re}\, p_m = 0 \tag{3.13}$$

The first two equations serve for the determination of the coordinates of the saddle point; the third signifies the condition of tangency of the zero level line at this point.

4. A concrete investigation of the character of instability in a compressible boundary layer was carried out for the first of the modes that arise additionally in comparison with the incompressible case, for M = 4.5 and T(0) = 4.44.

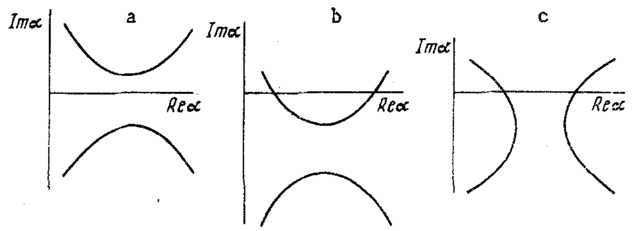

Fig. 1

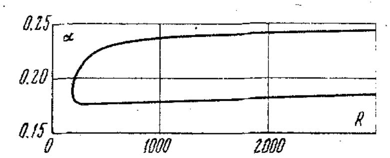

Fig. 2

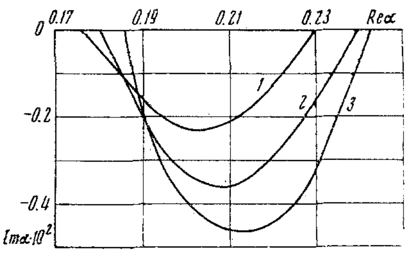

Fig. 3

For an incompressible boundary layer on a flat plate the problem concerned with the character of instability was considered in [8] with the assumption that the Reynolds numbers are large and, consequently, on both branches of the neutral curve the wave numbers are small. This allowed the authors [8] to use an asymptotic method. In the investigation it is shown that the boundary layer of Blasius is convectively unstable.

For a compressible boundary layer in the case of sufficiently large Mach numbers the most effective method of investigating the stability, apparently, is a numerical method, and more so, since in the case being considered in the present work both asymptotes of the neutral curve are nonviscous. A direct calculation of the level lines $\operatorname{Re} p = 0$ was carried out for a series of values of the Reynolds number. The calculations were carried out for two-dimensional disturbances by a method that is analogous to that used in [9]. In Fig. 2 we have presented the neutral curve, while in Fig. 3 we have shown the zero level lines for the Reynolds numbers 550, 1140, 3000, marked by the curves 1, 2, 3, respectively. From the results of the calculation we see that the instability arising when this mode is excited is convective.

5. The matrix $A(y)$ has the following nonzero components. By u, T, ρ, μ we have denoted the dimensionless (in relation to the corresponding values of the parameters of the incident flow) distributions of the longitudinal velocity, temperature, density, and viscosity in the boundary layer.

$$\sigma = p + iu, \quad \varphi = 1 / \gamma M^2 + 4\mu\sigma / 3R$$
$$a_{12} = 1, \quad a_{21} = \rho R\sigma / \mu + k^2, \quad a_{22} = -\mu' / \mu$$
$$a_{23} = i\alpha (R / (\gamma\mu M^2) + \sigma / 3), \quad a_{24} = \rho R u' / \mu + i\alpha (\ln (\rho^{1/3} / \mu))'$$
$$a_{27} = -i\alpha\sigma / (3T) - (\mu' u' / T')' / \mu, \quad a_{28} = -u'\mu' / (\mu T')$$
$$a_{31} = 2 i\alpha\mu (\ln (\rho^{2/3} / \mu))' / (R\varphi), \quad a_{32} = -i\alpha\mu / (R\varphi)$$
$$a_{33} = 4 \mu\sigma (\ln (\rho / \mu\sigma))' / (3R\varphi)$$
$$a_{34} = -\rho\sigma / \varphi - \mu k^2 / (R\varphi) + 4 \mu\rho' (\ln (\rho^2 / \mu \rho'))' / (3\rho R\varphi)$$
$$a_{35} = 2i\beta\mu (\ln (\rho^{3/2} / \mu))' / (R\varphi), \quad a_{36} = -i\beta\mu / (R\varphi)$$
$$a_{37} = i\alpha u'\mu' / (RT'\varphi) + 4\mu\sigma (\ln \mu\sigma)' / (3TR\varphi)$$
$$a_{38} = 4\mu\sigma / (3TR\varphi), \quad a_{41} = i\alpha, \quad a_{43} = -\sigma, \quad a_{44} = -\rho' / \rho$$
$$a_{45} = -i\beta, \quad a_{47} = \sigma / T, \quad a_{56} = 1, \quad a_{63} = i\beta (R / (\gamma\mu M^2) + \sigma / 3)$$
$$a_{64} = i\beta (\ln (\rho^{1/3} / \mu))', \quad a_{65} = R \rho\sigma / \mu + k^2$$
$$a_{66} = -\mu' / \mu, \quad a_{67} = -i\beta\sigma / (3T), \quad a_{78} = 1$$
$$a_{82} = -2 \Pr (\gamma - 1) M^2 u', \quad a_{83} = -(\gamma - 1) \Pr R\sigma / (\gamma\mu)$$
$$a_{84} = \Pr R\rho T' / \mu - 2i\alpha \Pr (\gamma - 1) M^2 u'$$
$$a_{87} = \Pr \rho R\sigma / \mu + k^2 - \mu'' / \mu - \Pr (\gamma - 1) M^2 u'^2\mu' / (\mu T')$$
$$a_{88} = -2\mu' / \mu$$

The nonzero components of the vector ξ are

$$\xi_2 = -(\rho R q_{\kappa 1}^{\circ} / \mu + i\alpha r_\kappa^\circ / 3\rho)$$

$$\xi_3 = \rho q_{\kappa 4}^{\circ} / \varphi + 4\,\mu r_\kappa^\circ (\ln (\mu r_\kappa^\circ / \rho^2))' / (3R\,\rho\varphi)$$

$$\xi_4 = r_\kappa^\circ / \rho, \quad \xi_6 = -\,R q_{k5}^\circ/\mu + i\beta r_k^\circ / (3\rho)$$

$$\xi_8 = -\,\sigma R \rho q^\circ_{k7} / \mu + \sigma R (\gamma - 1)\, q_{k3}^\circ / (\gamma\mu)$$

$$\left(\overset{\circ}{q}_{kn} = \frac{1}{(2\pi)^2} \int e^{-i\mathbf{kr}} q_n (t=0)\, d\mathbf{r}, \quad r_k^\circ / \rho = \overset{\circ}{q}_{k3} - \frac{\overset{\circ}{q}_{k7}}{T} \right)$$

6. The homogeneous equation (2.2) can be written in the form

$$q' = A(\infty)\, q + R(y) q \tag{6.1}$$

The matrices A(y), A (∞) are analytic in p in the region

$$D = \left\{ \mathrm{Re}\, p > -\frac{3R}{4\gamma M^2 \mu^*}, \quad |p| < \infty \right\} \qquad (\mu^* = \max_{0 \leqslant y < \infty} \mu)$$

It is required to show that there exist solutions ω_j such that

$$\lim_{y \to \infty} (w_j e^{-\lambda_j y} - p_j) = 0 \tag{6.2}$$

for $p \in D$. About R(y) we assume the following conditions of integratability [10]:

$$\int_0^\infty e^{\omega y} \| R(y) \| \, dy < \infty \tag{6.3}$$

for any $\infty > \omega \geq 0$, $p \in D$. The norm $\|R(y)\|$ is determined just as in [11].

We consider the integral equation

$$q = \psi_j - \int_y^\infty \Phi(y - \tau) R(\tau) q(\tau)\, d\tau \qquad (y \geqslant y_0 \geqslant 0) \tag{6.4}$$

By substitution we verify that if w_j satisfies (6.4); then

$$w_j' = A(y)\, w_j$$

The solution of Eq. (6.4) can be continued into the region $y < y_0$ as a solution of Eq. (6.1). By a method analogous to that used in [11], using (6.3) we can show that Eq. (6.1) has a solution w_j that satisfies the condition (6.2). Since the matrix dR/dp also satisfies the condition (6.3), we analogously can show that the derivative dw_j/dp exists and is unique for $p \in D_j$. Here $D_j = D - \Gamma_j$; Γ_j is a set of sections which are such that $\lambda_j(p)$ is a single-valued branch of the function $\lambda(p)$ in D_j.

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